14/04/2021 Part IV

Chp 1 – Poisson algebras

Mainly two families of Poisson algebras:

(1) < Symplectic smooth varieties.

(2) < "Almost commutative" filtered associative unital alg. (ex: $\mathcal{D}(G)$, $\mathcal{U}(\mathfrak{g})$)

§1. Ex arising from symplectic smooth varieties.

Let (X, ω) be a smooth symplectic affine variety, i.e. ω is a non-degenerate regular 2-form on X such that $d\omega = 0$

$\Rightarrow$ $\dim X$ is even.

Fact: $\mathcal{O}(X)$ has a natural Poisson structure

Examples:

(1) $X = \mathbb{C}^{2n}$ with coordinates $q_1, \dots, q_n, p_1, \dots, p_n$.

$\omega = dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$ symplectic structure.

(2) T^*X, X smooth, has canonical symplectic structure.

idea: we construct a 1-form on T^*X and we set $\omega = d\lambda$ ($\Rightarrow d\omega = 0$).

To construct λ: pick $x \in X$ and $\alpha \in T_x^* X$

$\lambda : T_\alpha(T^*X) \longrightarrow \mathbb{C}$.

Consider the canonical projection $\pi : T^*X \longrightarrow X$

the tangent map at α: $d_\alpha\pi : T_\alpha(T^*X) \longrightarrow T_x X$

let $\xi \in T_\alpha(T^*X)$. then define $\lambda(\xi)$

$$\begin{array}{ccccc} T_\alpha(T^*X) & \xrightarrow{d_\alpha\pi} & T_x X & \xrightarrow{\alpha} & \mathbb{C}. \\ \xi & \longmapsto & d_\alpha\pi(\xi) & \longmapsto & \langle \alpha, d_\alpha\pi(\xi)\rangle \end{array}$$

$\omega = d\lambda$

To see the non-degenerate character.

In coordinates: let $(q_1 \dots q_n)$ local coord on X, and $p_1 \dots p_n$ additional local coord in T^*X.

Write $T^*X \ni \alpha = (\underbrace{q_1(\alpha), \dots, q_n(\alpha)}_{x \in X}, \underbrace{p_1(\alpha), \dots, p_n(\alpha)}_{T_x X})$

$\xi \in T_\alpha(T^*X)$ has the form $\xi = \sum_i b_i \frac{\partial}{\partial p_i} + \sum_i c_i \frac{\partial}{\partial q_i}$ $\quad b_i, c_i \in \mathbb{C}$.

thus: $x = \pi(\alpha) = (q_1(\alpha), \dots, q_n(\alpha))$ $\quad d_\alpha\pi : T_\alpha(T^*X) \longrightarrow T_x X$ is given by

$$\xi \longmapsto d_\alpha\pi(\xi) = \sum c_i \frac{\partial}{\partial q_i}$$

We see that $\lambda(\xi) = \langle \alpha, d_\alpha\pi(\xi)\rangle = \sum_i p_i(\alpha)\, c_i$

<u>concl</u>: $\lambda = \sum p_i\, dq_i$ and $\omega = d\lambda = \sum dp_i \wedge dq_i$

| whence ω gives a sympl. structure on T^*X.

(3) G is an algebraic Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$

$G \curvearrowright G$ by $g \cdot h = g h g^{-1} = \mathrm{Int}(g)(h)$

$g \in G$ $\quad \mathrm{Int}(g) : G \longrightarrow G$

$$\mathrm{Ad}(g) = d_e \mathrm{Int}(g) : \underset{\simeq \mathfrak{g}}{T_e G} \longrightarrow \underset{\simeq \mathfrak{g}}{T_e G}.$$

$$\mathrm{Ad} : G \longrightarrow \mathrm{End}(\mathfrak{g}), \quad g \in G \longmapsto \mathrm{Ad}(g)$$

$\rightsquigarrow \boxed{G \curvearrowright \mathfrak{g}}$.

$$d_e \mathrm{Ad} : T_e G = \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \quad x \longmapsto \mathrm{ad}\, x : y \mapsto [x, y].$$

G acts on $\mathfrak{g}$ and on $\mathfrak{g}^*$

$$g \cdot \xi = \mathrm{Ad}^*(g)\xi : \quad x \in \mathfrak{g} \longmapsto \xi(\mathrm{Ad}(g^{-1})x)$$

<u>Rem</u>: $\mathrm{Ad}^* : G \longrightarrow \mathrm{End}(\mathfrak{g}^*) \quad \rightsquigarrow \quad \mathrm{ad}^* : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}^*)$

where: $(\mathrm{ad}^* x)(\xi) : v \in \mathfrak{g} \longmapsto -\xi((\mathrm{ad}\, x) v) = -\xi([x, v])$

<u>Prop</u>: Any coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ ($G \cdot \xi$, $\xi \in \mathfrak{g}^*$) has a natural symplectic structure (Kostant–Kirillov–Souriau)

<u>proof</u>: $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$

We have to produce a skew-symmetric form on $T_\alpha \mathbb{O}$.

$\mathbb{O} \simeq G/G^\alpha$, where G^α is the isotropy group (stabilizer) of α in G

therefore $T_\alpha \mathbb{O} = T_\alpha(G/G^\alpha) = \mathfrak{g}/\mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha = \mathrm{Lie}(G^\alpha) = \{x \in \mathfrak{g} : (\mathrm{ad}^* x)\alpha = 0\}$

$\rightsquigarrow$ we have to define a skew-symmetric 2-form on $\mathfrak{g}/\mathfrak{g}^\alpha$.

first define : $\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $\omega_\alpha(x,y) = \alpha([x,y])$

this form descends to $\mathfrak{g}/\mathfrak{g}^\alpha$

$g \in G^\alpha \iff (\mathrm{Ad}^* g)(\alpha) = \alpha$

Differentiating at $g = e$, we obtain that $(\mathrm{ad}^* x)\alpha = 0 \iff x \in \mathfrak{g}^\alpha$

$$(\mathrm{ad}^* x)(\alpha)\, y = -\alpha(\underbrace{[x,y]}_{=\omega_\alpha(x,y)})$$

therefore :

$$\omega_\alpha(x,y) = 0 \quad \forall y \in \mathfrak{g} \iff x \in \mathfrak{g}^\alpha$$

thus : $\omega_\alpha : \mathfrak{g}/\mathfrak{g}^\alpha \times \mathfrak{g}/\mathfrak{g}^\alpha \longrightarrow \mathbb{C}$ is well-defined and non-degenerate.

<u>claim</u>: $d\omega = 0$.

~~Proof~~ : Cartan formula:

$$d\omega(\xi_1,\xi_2,\xi_3) = \xi_1 \cdot \omega(\xi_2,\xi_3) + \xi_2 \cdot \omega(\xi_3,\xi_1) + \xi_3 \cdot \omega(\xi_1,\xi_2)$$
$$- \omega([\xi_1,\xi_2],\xi_3) - \omega([\xi_2,\xi_3],\xi_1) - \omega([\xi_1,\xi_3],\xi_1).$$

Any $x \in \mathfrak{g}$ gives rise to a vector field ξ_x on $\mathcal{O}$.

ξ_x, $x \in \mathfrak{g}$, span $T_\alpha \mathcal{O}$ to each $\alpha \in \mathcal{O}$.

Hence to show that $d\omega = 0$, it suffices to check that

$$d\omega(\xi_x,\xi_y,\xi_z) = 0 \quad \forall x,y,z \in \mathfrak{g}$$

$\omega(\xi_x,\xi_y)(\alpha) = \alpha([\xi_x,\xi_y])$ $\qquad (\xi_x \cdot y)(\alpha) = \alpha([x,y])$ $\quad x,y \in \mathfrak{g}$

+ Jacobi $\longrightarrow$ get the statement. $\boxed{}$

Poisson structure on $\mathcal{O}(X)$ for a sympl. smooth variety (X,ω)

$\omega \rightsquigarrow \underset{\text{canonical}}{TX \simeq T^*X}$: $T_xX \longrightarrow T_x^*X,\ \xi \longmapsto (\eta \longmapsto \omega_x(\xi\,\eta))$

Define a linear map:

$\mathcal{O}(X) \longrightarrow \{\text{vector fields on } X\},\ f \longmapsto \xi_f$ is defined by :

$\omega(\cdot, \xi_f) = df$ $\quad (x \in X,\ \eta \in T_xX\ ,\ \omega_x(\eta, \xi_f) = d_xf(\eta))$

We define a bracket on $\mathcal{O}(X)$ by :

$\{f,g\} := \omega(\xi_f, \xi_g)$ $(= d_g(\xi_f) =: \xi_f \cdot g)$

Def : A vector field ξ is called symplectic if it preserves the symplectic form, i.e:

$L_\xi \omega = 0$, where L_ξ is the Lie derivative

L_ξ acts on Ω_X $\quad \boxed{L_\xi(\alpha) = i_\xi d\alpha + d\, i_\xi \alpha}$

$d : \{k\text{-forms}\} \longrightarrow \{(k+1)\text{-forms}\}$

$i : \{k\text{-forms}\} \longrightarrow \{(k-1)\text{-forms}\}$

$(i_\xi \omega)(\xi_1, \dots, \xi_{k-1}) = \omega(\xi, \xi_1, \dots, \xi_{k-1})$

In particular if α : $i_\xi(\alpha) = \alpha(\xi) = \langle \alpha, \xi \rangle$

Lemma : $\forall f \in \mathcal{O}(X)$, ξ_f is symplectic : $\underset{i_{\xi_f} d\omega + d i_{\xi_f}\omega}{\overset{\shortparallel}{L_{\xi_f}}} \omega = 0$

proof : (1) ω is closed, $d\omega = 0$ (hyp)

(2) $d i_{\xi_f} \omega = -d(df) = 0$. $\qquad \omega(\cdot, \xi_f) = df \qquad -df = i_{\xi_f}\omega$

Hence $L_{\xi_f}\omega = 0$

Prop: We have a Lie homom:

$$(\mathcal{O}(X), \{,\}) \longrightarrow \{\text{symplectic vector fields on } X\}$$

Lie algebra.

$$f \longmapsto \xi_f .$$

proof: We have to show $[\xi_f, \xi_g] = \xi_{\{f,g\}}$

Then use properties of Lie derivative ... $\boxed{}$

Thm: $(\mathcal{O}(X), \{,\})$ is a Poisson algebra. (X sympl. smooth (affine) variety)

proof: • Jacobi identity:

By the proposition: $[\xi_f, \xi_g] . h = \xi_{\{f,g\}} h = \{\{f,g\}, h\}$ (1)

$[\xi_f, \xi_g] h = \xi_f \xi_g h - \xi_g \xi_f h = \{f, \{g,h\}\} - \{g, \{f,h\}\}$ (2)

(1) − (2) $\Longrightarrow$ Jacobi.

. Leibniz: clear since $g \longmapsto \xi_f g$, $\forall f \in \mathcal{O}(X)$, is a derivation of $\mathcal{O}(X)$. $\boxed{}$

Ex: $(\mathcal{O}(T^*X), \{,\})$ and $(\mathcal{O}(G.\xi), \{,\})$ are Poisson

$\xi \in \mathfrak{g}^*$.

§2 - Examples arising from almost commutative assoc. alg.

Let A be a filtered assoc. (non-comm). alg. with unit

$$\mathbb{C} \subset A_0 \subset A_1 \subset \dots \qquad \bigcup A_i = A. \quad , \quad A_i . A_j \subset A_{i+j}$$

$$\bar{A} = \operatorname{gr} A = \bigoplus_{p \geq 0} A_p / A_{p-1} \quad , \quad A_{-1} = \{0\}$$

the product in A gives rise to a well-defined product on $\bar{A}$:

$$A_i / A_{i-1} \times A_j / A_{j-1} \longrightarrow A_{i+j} / A_{i+j-1}$$

this makes $\bar{A}$ an assoc. alg structure

Def: A is almost-commutative i.e.: $\bar{A}$ is comm.

Prop: If A is almost-comm., then $\bar{A} = \operatorname{gr} A$ has a natural Poisson structure.

$$\{\cdot,\cdot\} : A_i / A_{i-1} \times A_j / A_{j-1} \longrightarrow A_{i+j-1} / A_{i+j-2}$$

$$(a_i + A_{i-1} , a_j + A_{j-1}) \longmapsto \underbrace{a_i a_j - a_j a_i}_{\in A_{i+j-1}} + A_{i+j-2} .$$

Proof: well-defined ✓

Jacobi + Leibniz: are satisfied for $\{a_i, a_j\} := a_i a_j - a_j a_i$

$\{xy, z\} = x\{y,z\} + \{x,z\}y$.

$\rightarrow$ so in $\bar{A} = \operatorname{gr} A$ we get the statement. ∎

Ex: $\mathcal{U}(\mathfrak{g})$, $\mathcal{D}(C)$, $Zhu(V)$ where V is a graded VA.

Namely: $\operatorname{gr} \mathcal{U}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathfrak{g}^*$ is Poisson

$\operatorname{gr} \mathcal{D}(X) \simeq \mathcal{O}(T^*X) \longrightarrow T^*X$ Poisson

Rem: if $\forall a_i \in A_i,\ g_j \in A_j$, $a_i a_j - a_j a_i \in A_{i+j-2}$

$$\{,\} : A_i/A_{i-1} \times A_j/A_{j-1} \longrightarrow A_{i+j-2}/A_{i+j-3}$$
$$a_i,\ a_j \longmapsto a_i a_j - a_j a_i + A_{i+j-3}.$$

Because A is non commutative, there is always a nontrivial Poisson structure in this way.

Ex: (1) $\operatorname{gr} \mathcal{U}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*] \rightsquigarrow (\underset{S(\mathfrak{g})}{\mathbb{C}[\mathfrak{g}^*]}, \{,\})$ Poisson

In particular, $\mathfrak{g}^*$ is a Poisson variety.

$f, g \in \mathbb{C}[\mathfrak{g}^*]$

$(*)$ $\{f, g\}(\xi) = \langle \xi, [d_\xi f, d_\xi g] \rangle.$ $\quad f : \mathfrak{g}^* \longrightarrow \mathbb{C}$

$\xi \in \mathfrak{g}^*$ $\quad$ viewed as element of $\mathfrak{g}$ $\quad d_\xi f : \mathfrak{g}^* \longrightarrow \mathbb{C}$ linear

$d_\xi f \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$.

Check: $\mathfrak{g} \subset S(\mathfrak{g}) = \operatorname{gr} \mathcal{U}(\mathfrak{g})$

$xy - yx = [x,y]$ ✓ $\longrightarrow$ $(*)$ for $f = x$, $g = y$.

Enough to check on degree one generators.

$\mathbb{O} = G.\xi \qquad \mathcal{O}(G.\xi)$ Poisson $\quad \{\ \}_{symp}$.

$f, g \in \mathcal{O}(\mathfrak{g}^*) = \mathbb{C}[\mathfrak{g}^*] \qquad \{f, g\}$

$$\{f,g\}|_{\mathbb{O}}(\xi) \underset{\substack{?\\ \text{they are the same.}}}{=} \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symp}(\xi)$$

proof: enough to show the statement for elements of $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$, $\alpha \in \mathbb{O} \subset \mathfrak{g}^*$

$[x,y](\alpha) = \alpha([x,y]) = (\operatorname{ad} x)(y)|_\alpha = (\xi_x . y)|_\alpha = \{x|_{\mathbb{O}}, y|_{\mathbb{O}}\}_{symplectic}(\alpha)$ $\quad$ 4

(2) $\mathrm{gr}\, \mathcal{D}(G) \simeq \mathbb{C}[T^*G]$

<u>thm</u>: Poisson structure on $\mathcal{O}(T^*\overset{\times}{G}) = \mathrm{gr}\, \mathcal{D}(\overset{\times}{G})$ is the same as the one coming from the sympl. structure on T^*G (holds for any smooth alg. variety X).

<u>proof</u>: $\mathcal{O}(T^*X)$ is gen. by the subalg $\mathcal{O}(X) \subset \mathcal{O}(T^*X)$ ($\pi : T^*X \to X$) and by the space of functions that are linear along the fibers (ie: symbols of the vector fields on X.

Due to Leibniz, enough to check the Poisson bracket on these generators.

* those involving $\mathcal{O}(X)$: easy.
* ξ, η vector field on X, $\xi\eta - \eta\xi = [\xi, \eta]$ ... $\boxtimes$

§3 – Stratification into symplectic leaves

Let X be an affine algebraic variety (reduced), $X = \operatorname{Spec} A$, A reduced.

Assume that $(A, \{,\})$ Poisson (X is a Poisson variety)

Ex: $\mathfrak{g}^*$, T^*G, $X_V = (\operatorname{Spec} R_V)_{red}$, $(\operatorname{Spec} \operatorname{gr} Z_{\hbar} V)_{red}$, etc...

Assume first X is smooth.

known fact:

X Poisson $\iff \exists$ closed regular 2-form $\overset{w}{V}$ on X (not necessarily non-degenerate)

ex: on $\mathfrak{g}^*$: $\omega_\alpha : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$, $(x,y) \longmapsto \alpha([x,y])$.

$\alpha \in \mathfrak{g}^*$ $\qquad \mathbb{O} = G\cdot\alpha \qquad \mathbb{O} = G/G^\alpha \qquad T_\alpha \mathbb{O} = \mathfrak{g}/\mathfrak{g}^\alpha$.

$x \in X$: the symplectic leaf $\mathcal{L}(x)$ containing x is the maximal connected complex analytic manifold in X st $\{,\}$ is non-degenerate at any pt of $\mathcal{L}(x)$.

ie: $\mathcal{L}(x)$ is the max sympl. variety containing x.

More analytic X smooth, $x, y \in X$

$x \sim y \iff \exists\ \sigma : B(\varepsilon) \xrightarrow{\text{ball around } 0} \mathbb{C}$, $\sigma(0) = x$, $\sigma(z) = y$ for some $z \in B(\varepsilon)$.

and σ is an integral curve of some Hamiltonian $\{H, \cdot\}$, $H \in A$.

ie: $\forall f \in A$, $\frac{d}{dz}(f\circ\sigma) = \{H, f\}\circ\sigma$.

$\mathcal{L}(x) = \{y \in X : y \sim x\}$.

If X is not smooth

$X_0 := X_{reg}$: the smooth locus $\quad X_0 = \bigsqcup_{x} \mathcal{L}(x)$ from the smooth case

$\widetilde{X_1} = X \setminus X_{reg}$. $\quad X_1 = (\widetilde{X_1})_{reg} \leadsto$

$\vdots$

$X = \bigsqcup_{k} X_k$, each X_k is smooth variety. $\quad X_k \leadsto$ decomposes into symplectic leaves.

$$X = \bigsqcup_{x} \mathcal{L}(x)$$